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CITATION:

Tsugawa, Kotaro. On the coupled system of nonlinear wave equations with different propagation speeds in two space dimensions (Non linear evolution equation and its applications). 数理解析研究所講究録 2000, 1135: 85-90

ISSUE DATE:

2000-04

URL:

<http://hdl.handle.net/2433/63763>

RIGHT:

On the coupled system of nonlinear wave equations with different propagation speeds in two space dimensions

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1 Introduction and Main results

In the present paper, we treat the coupled system of wave equations with different propagation speeds:

$$(\partial_t^2 - \Delta)f = F(f, \partial f, g, \partial g), \quad x \in \mathbb{R}^n, t \in \mathbb{R}, \quad (1.1)$$

$$(\partial_t^2 - s^2 \Delta)g = G(f, \partial f, g, \partial g), \quad x \in \mathbb{R}^n, t \in \mathbb{R}, \quad (1.2)$$

$$f(x, 0) = f_0(x) \in H^a, \quad \partial_t f(x, 0) = f_1(x) \in H^{a-1}, \quad x \in \mathbb{R}^n, \quad (1.3)$$

$$g(x, 0) = g_0(x) \in H^a, \quad \partial_t g(x, 0) = g_1(x) \in H^{a-1}, \quad x \in \mathbb{R}^n, \quad (1.4)$$

where $\partial = \partial_{x_j}$ ($1 \leq j \leq n$) or ∂_t and s is a propagation speed of (1.2) with $s > 1$. The nonlinear terms are as follows:

$$F = \sum_{j=1}^3 \alpha_j F_j, \quad \alpha_j \in \mathbb{C},$$

$$G = \sum_{j=1}^3 \beta_j G_j, \quad \beta_j \in \mathbb{C},$$

$$F_1 = g \partial g, \quad F_2 = f \partial g, \quad F_3 = g \partial f,$$

$$G_1 = f \partial f, \quad G_2 = f \partial g, \quad G_3 = g \partial f.$$

Our aim is to prove the time local well-posedness with the low regularity initial data. Physically, this system describes the Klein-Gordon-Zakharov equations (K-G-Z) and the coupled system of complex scalar field and Maxwell equations (C-M). We can derive the time local well-posedness of (K-G-Z) and (C-M) from the time local well-posedness of this system.

In the case of $n \geq 4$, we can prove the time local well-posedness with $a \geq (n-1)/2$ by the Strichartz estimate. This proof is indepent of the difference of the speeds. In the case of $n = 3$, we can prove the time local well-posedness with $a > 1$ by the Strichartz estimate. To prove the time local well-posedness with $a = 1$ in this argument, we need the limiting case of the Strichartz estimate, which fails. But, Ozawa, Tsutaya and Tsutsumi[5] proved the time local well-posedness in the case of $F = F_1, G = G_3$ with $a = 1$ by using the difference of speeds and Fourier restriction norm method. By this result and the energy conservation, they proved the time global well-posedness of (K-G-Z). By the same argument, T[7] proved the time local well-posedness in the case of $F = F_2, F = F_3, G = G_1$ with $a = 1$. By this result and the energy conservation, we had the time global well-posedness of (C-M).

Fourier restriction norm method was developed by Bourgain [1] and [2] to study the nonlinear Schrödinger equation and the KdV equation, and it was improved for the one dimensional case by Kenig, Ponce and Vega[4]. The related method was developed by Klainerman and Machedon [3] for the nonlinear wave equations.

In the case of $n = 2$, it seem to be difficult to prove the time local well-posedness with $a < 3/4$ by the Strichartz estimate. But, in the present paper, we have the time local well-posedness with $a > 1/2$ by using the difference of speeds and Fourier restriction norm method.

Before we state the theorem, we give several notations. For a function $u(t, x)$, we denote by $\tilde{u}(\tau, \xi)$ the Fourier transform in both x and t variables of u . For $a, b \in \mathbb{R}, s > 0$ and $l = +$ or $-$, we define the spaces $X_{s,l}^{a,b}$ as follows:

$$X_{s,l}^{a,b} = \{u \in \mathcal{S}'(\mathbb{R}^3) \mid \|u\|_{X_{s,l}^{a,b}} < \infty\}$$

$$\|u\|_{X_{s,l}^{a,b}} = \| \langle \xi \rangle^a P_{s,l}^b(\tau, \xi) \tilde{u} \|$$

where $P_{s,l}(\tau, \xi) = (1 + |\tau + sl|\xi|)|$, $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$ and $\|\cdot\| = \|\cdot\|_{L_{\tau,\xi}^2}$. For $T > 0$, we denote the cut function $\chi(t), \chi_T(t) \in C_0^\infty$ as follows:

$$\chi(t) = \begin{cases} 1 & \text{for } |t| \leq 1/2, \\ 0 & \text{for } |t| > 1, \end{cases}$$

$$\chi_T(t) = \chi(t/T).$$

For $s > 0$, we define $W_{s,\pm}(t) = e^{\mp i s t \omega}$, where $\omega = \sqrt{1 - \Delta}$. We put

$$\langle f, g \rangle = \int_{\mathbb{R}^3} f(t, x) \overline{g(t, x)} dt dx.$$

Theorem 1.1. *Let $s > 1$ or $1 > s > 0, a > 1/2$ and $2a - 1/2 > b > 1/2$, then there exist $T > 0$ and problem (1.1)-(1.4) has time local unique solution satisfying*

$$f, g \in C([-T, T] : H^a(\mathbb{R}^2)) \cap C^1([-T, T] : H^{a-1}(\mathbb{R}^2)), \quad (1.5)$$

$$\begin{aligned}\chi_T(t)(f \pm i\omega^{-1}\partial_t f) &\in X_{1,\pm}^{a,b}, \\ \chi_T(t)(g \pm i(s\omega)^{-1}\partial_t g) &\in X_{s,\pm}^{a,b}.\end{aligned}$$

Furthermore, this solution depends continuously on initial datas in the topology of (1.5).

2 The proof of the Theorem

We first put

$$\begin{aligned}f_{\pm} &= f \pm i\omega^{-1}\partial_t f, \\ g_{\pm} &= g \pm i(s\omega)^{-1}\partial_t g.\end{aligned}$$

Then, (1.1)-(1.4) are rewritten as follows:

$$(i\partial_t \mp D)f_{\pm} = \mp\omega^{-1}F \mp (D - \omega)f_{\pm}, \quad (2.1)$$

$$(i\partial_t \mp sD)g_{\pm} = \mp(s\omega)^{-1}G \mp s(D - \omega)g_{\pm}, \quad (2.2)$$

$$f_{\pm}(0) = f_{\pm 0}, \quad g_{\pm}(0) = g_{\pm 0}, \quad (2.3)$$

where

$$\begin{aligned}f_{\pm 0} &= f_0 \pm i\omega^{-1}f_1 \in H^a, \\ g_{\pm 0} &= g_0 \pm i(s\omega)^{-1}g_1 \in H^a.\end{aligned}$$

We try to solve (2.1)-(2.3) locally in time. For that purpose, we consider the following integral equations associated with (2.1)-(2.3):

$$f_{\pm}(t) = \chi(t)W_{1,\pm}(t)f_{\pm 0} \mp i\chi_T(t) \int_0^t W_{1,\pm}(t-s)\{\omega^{-1}F + (D - \omega)f_{\pm}\}ds, \quad (2.4)$$

$$g_{\pm}(t) = \chi(t)W_{s,\pm}(t)g_{\pm 0} \mp i\chi_T(t) \int_0^t W_{s,\pm}(t-s)\{\omega^{-1}G + (D - \omega)g_{\pm}\}ds. \quad (2.5)$$

If we try to apply the Fourier restriction norm method to (2.4)-(2.5), we have only to prove the following estimates:

$$\|F_1\|_{X_{1,l}^{a-1,b-1+\epsilon}} \leq C\Sigma_{j,k}\|g_j\|_{X_{s,j}^{a,b}}\|g_k\|_{X_{s,k}^{a,b}}, \quad (2.6)$$

$$\|F_2\|_{X_{1,l}^{a-1,b-1+\epsilon}} \leq C\Sigma_{j,k}\|f_j\|_{X_{1,j}^{a,b}}\|g_k\|_{X_{s,k}^{a,b}}, \quad (2.7)$$

$$\|F_3\|_{X_{1,l}^{a-1,b-1+\epsilon}} \leq C\Sigma_{j,k}\|f_j\|_{X_{1,j}^{a,b}}\|g_k\|_{X_{s,k}^{a,b}}, \quad (2.8)$$

$$\|G_1\|_{X_{s,l}^{a-1,b-1+\epsilon}} \leq C\Sigma_{j,k}\|f_j\|_{X_{1,j}^{a,b}}\|f_k\|_{X_{1,k}^{a,b}}, \quad (2.9)$$

$$\|G_2\|_{X_{s,l}^{a-1,b-1+\epsilon}} \leq C\Sigma_{j,k}\|f_j\|_{X_{1,j}^{a,b}}\|g_k\|_{X_{s,k}^{a,b}}, \quad (2.10)$$

$$\|G_3\|_{X_{s,l}^{a-1,b-1+\epsilon}} \leq C\Sigma_{j,k}\|f_j\|_{X_{1,j}^{a,b}}\|g_k\|_{X_{s,k}^{a,b}}, \quad (2.11)$$

where $a > 1/2$, $2a - 1/2 > b > 1/2$ and $\epsilon > 0$ which is sufficiently small and j, k and l denote either of $+$ or $-$ sign. Without loss of generality, we can assume \tilde{f}_j and $\tilde{g}_k > 0$. Here, we note that

$$\begin{aligned} f &= 1/2(f_+ + f_-), \\ \partial_t f &= \frac{\omega}{2i}(f_+ - f_-), \\ g &= 1/2(g_+ + g_-), \\ \partial_t g &= \frac{s\omega}{2i}(g_+ - g_-). \end{aligned}$$

Therefore, the left hand side of (2.6) is bounded by

$$\Sigma_{j,k} \|g_j \omega g_k\|_{X_{1,l}^{a-1,b-1+\epsilon}}. \quad (2.12)$$

To prove (2.6), we have only to prove

$$\|g_j \omega g_k\|_{X_{1,l}^{a-1,b-1+\epsilon}} \leq C \|g_j\|_{X_{s,j}^{a,b}} \|g_k\|_{X_{s,k}^{a,b}},$$

which is equivalent to

$$\langle g_j g_k, h \rangle \leq C \|g_j\|_{X_{s,j}^{a,b}} \|g_k\|_{X_{s,k}^{a-1,b}} \|h\|_{X_{1,l}^{1-a,1-b-\epsilon}}$$

by duality argument. We obtain this inequality by interpolating between (2.13) and (2.14). In the same manner, we obtain (2.7)-(2.11) from Proposition 2.1.

Proposition 2.1. *Assume that $a > 1/2$, $b > 1/4$, $4a + 2b > 3$ and $s > 1$ or $0 < s < 1$. Then the following inequalities hold.*

$$|\langle f, gh \rangle| \leq C \|f\|_{X_{1,j}^{-a,b}} \|g\|_{X_{s,k}^{a,b}} \|h\|_{X_{1,l}^{a,b}} \quad (2.13)$$

$$|\langle f, gh \rangle| \leq C \|f\|_{X_{s,j}^{-a,b}} \|g\|_{X_{1,k}^{a,b}} \|h\|_{X_{1,l}^{a,b}} \quad (2.14)$$

where j, k and l denote either of $+$ or $-$ sign.

Remark 2.1. This inequalities hold with $a = 1/2$, $b > 1/2$. But, because of $b > 1/2$, we can't apply Proposition 2.1 to (1.1)-(1.4).

Before we prove Proposition 2.1, we mention an essential lemma.

Lemma 2.1. *Assume that $a > 1/2$, $b > 1/4$, $4a + 2b > 3$ and $s > 1$ or $0 < s < 1$. Then, there is a positive constant C and the following inequalities hold.*

$$\sup_{\tau, \xi} \langle \xi \rangle^{2a} P_{1,j}^{-2b}(\tau, \xi) \left(\langle \xi \rangle^{-2a} P_{s,k}^{-2b}(\tau, \xi) *_{\tau, \xi} \langle \xi \rangle^{-2a} P_{1,l}^{-2b}(\tau, \xi) \right) < C$$

where j, k and l denote either of $+$ or $-$ sign.

Proof of Proposition 2.1. Without loss of generality, we can assume \tilde{f}, \tilde{g} and $\tilde{h} > 0$. We first prove (2.13). By the duality argument, (2.13) is equivalent to

$$\| \langle \xi \rangle^a P_{1,j}^{-b}(\tau, \xi) \tilde{g} \tilde{h} \|_{L_{\tau,\xi}} \leq C \| \langle \xi \rangle^a P_{s,k}^b(\tau, \xi) \tilde{g} \| \| \langle \xi \rangle^a P_{s,l}^b(\tau, \xi) \tilde{h} \|,$$

which is equivalent to

$$\| \langle \xi \rangle^a P_{1,j}^b(\tau, \xi) (\langle \xi \rangle^{-a} P_{s,k}^{-b}(\tau, \xi) \tilde{g} *_{\tau,\xi} \langle \xi \rangle^{-a} P_{s,l}^{-b}(\tau, \xi) \tilde{h}) \| \leq C \| \tilde{g} \|^2 \| \tilde{h} \|^2. \quad (2.15)$$

By the Schwartz's inequality and Lemma 2.1, the left hand side of (2.15) is bounded by

$$\begin{aligned} \int_{\mathbb{R}^3} \langle \xi \rangle^{2a} P_{1,j}^{2b}(\tau, \xi) (\langle \xi \rangle^{-2a} P_{s,k}^{-2b}(\tau, \xi) *_{\tau,\xi} \langle \xi \rangle^{-2a} P_{s,l}^{-2b}(\tau, \xi)) (\tilde{g}^2 *_{\tau,\xi} \tilde{h}^2) d\tau d\xi \\ \leq C \int_{\mathbb{R}^3} \tilde{g}^2 *_{\tau,\xi} \tilde{h}^2 d\tau d\xi \\ \leq C \| \tilde{g}^2 \|_{L_{\tau,\xi}^1} \| \tilde{h}^2 \|_{L_{\tau,\xi}^1} \\ \leq C \| \tilde{g} \| \| \tilde{h} \|. \end{aligned}$$

We next prove (2.14). From (2.13), we have

$$\begin{aligned} |\langle f, gh \rangle| &= | \langle \omega^{-2a} f, \omega^{2a} (gh) \rangle | \\ &\leq | \langle \omega^{2a} g, (\omega^{-2a} f) \bar{h} \rangle | + | \langle \omega^{2a} h, (\omega^{-2a} f) \bar{g} \rangle | \\ &\leq C \| g \|_{X_{1,j}^{a,b}} \| f \|_{X_{s,k}^{-a,b}} \| h \|_{X_{1,l}^{a,b}}. \end{aligned}$$

□

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